Unabridged[1] preface

> "Il doit bien se présenter des problèmes de Physique mathématique pour lesquels les causes physiques de régularité ne suffisent pas à justifier les hypothèses de régularité faites lors de la mise en équation."[2]
>
> Jean Leray

This book is devoted to classical techniques in elliptic partial differential equations (PDEs), involving solutions that are not expected to be smooth. Some of the topics that are developed are: regularity theory, maximum principles, Perron–Remak method, sub- and supersolutions, and removable singularities. They rely on tools from measure theory, functional analysis, and Sobolev spaces [52, 123, 132, 158, 344].

The goal is to investigate the *linear Dirichlet problem* involving the Laplacian:

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{DP}$$

for an arbitrary *finite Borel measure* μ; the semilinear counterpart of problem (DP) is also considered. The quantity $\mu(A)$ can be interpreted as the mass or total charge contained in a subset $A \subset \Omega$.

An example of solution is provided by the classical Green's function with a Dirac mass $\mu = \delta_a$. For any smooth bounded open set Ω, this problem has a unique solution for any measure μ. By a solution, we mean a summable function that verifies the equation against smooth test functions vanishing on the boundary $\partial\Omega$. This weak formulation implicitly encodes the zero boundary condition.

We have gathered several elegant proofs which are mostly available in the literature, but that are not necessarily widespread within the mathematical community. A surprising example is the simple argument leading to the fractional

---

[1] This unabridged version includes a detailed description of the chapter contents and will be included in a future edition of the book.

[2] "There must be problems in mathematical physics for which the physical regularity causes are not enough to justify the regularity assumptions needed to derive the equation."

Sobolev imbedding. We also explain the connection between trace inequalities and the strong approximation of diffuse measures.

The reader should feel free to choose a topic according to his/her own interests. The chapters have been conceived to be as independent as possible. We begin with some brief historical perspective in **Chapter 0** to explain how the study of classical potential theory evolved since the 18^{th} century to more recent nonlinear problems dealing with measure data.

Chapters 1–3 present some introductory material intended to familiarize the reader with the notation and basic notions that lead one to the weak formulation of the Dirichlet problem (DP). **Chapter 1** starts with the classical Poisson equation

$$-\Delta u = \mu \quad \text{in } \Omega, \tag{PE}$$

where we recall some properties of smooth harmonic and superharmonic functions in connection with monotonicity formulas and maximum principles.

In **Chapter 2**, we consider solutions of the Poisson equation (PE) involving measure data μ, where the equation is now understood in the sense of distributions. We show using the Riesz representation theorem that every weak superharmonic function satisfies the Poisson equation for some nonnegative measure. A short review of properties of finite measures is presented in the beginning of the chapter.

Test functions with compact support are unable to detect boundary values of a solution of the Poisson equation (PE). In **Chapter 3** we explain the weak formulation of the Dirichlet problem (DP) in the spirit of the work of Littman, Stampacchia and Weinberger [211], and prove the existence and uniqueness of solutions. In this approach, there is little difference of whether the density μ is a summable function or a finite measure. In contrast, as it was first pointed out by Bénilan and Brezis [23], the semilinear counterpart

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{SDP}$$

need not have a solution when μ is a Dirac mass, depending on the growth of g at infinity.

Chapters 4–6 address fundamental questions concerning the existence, regularity and uniqueness of solutions. Typical solutions of (DP) and (SDP) with measure data cannot be obtained by variational methods through minimization because the energy functionals that are associated to them need not be bounded from below. Nevertheless, for better data μ – for instance in $L^2(\Omega)$ – we implement the variational approach in **Chapter 4** using minimizing sequences in the Sobolev space $W_0^{1,2}(\Omega)$. The more subtle question of whether the Euler–Lagrange equation holds is also addressed. These variational solutions are important since they are the building blocks upon which solutions with measure data can be constructed by approximation.

We recall the definition and some basic properties of the Sobolev spaces $W_0^{1,q}(\Omega)$. Then, in **Chapter 5**, we prove the Sobolev regularity of the solutions of (DP), which physically yields the existence of a force field $F = -\nabla u$ for any finite measure μ. We also present an elegant proof by Boccardo and Gallouët [29] of the embedding of ∇u in the weak $L^{\frac{N}{N-1}}$ space based on Stampacchia's truncation argument.

In **Chapter 6**, we investigate maximum principles adapted to the formalism of weak solutions. We also analyze Kato's inequality, which allows one to compare solutions of the semilinear problem (SDP). This study is pursued in **Chapter 7**, where we prove the finiteness of the measure Δu^+ and the existence of the weak normal derivative $\frac{\partial u}{\partial n}$ on $\partial\Omega$. We explain how these properties are related to Poincaré's balayage method.

From this point of the book on, the notion of capacity is needed to investigate deeper properties of the Laplacian. Why are capacities so important? Because they help one to identify sets which are possibly undetected by solutions of (DP). This is analogous to the fact that sets of zero Lebesgue measure are irrelevant for summable (Lebesgue) functions. But the better the function is, the smaller the exceptional set is, and such a smallness information can be typically described by some capacity.

This aspect is carefully exploited in **Chapter 8**, where we apply some tools from geometric measure theory to identify the correct capacity in various situations. We have gathered in **Chapter 9** the more technical aspects related to maximal inequalities. As a first example of how to manipulate the notion of precise representative, we prove the general formulation of Kato's inequality when Δu is a measure.

To get some intuition of how big sets of zero capacity are, one can use a more geometric concept to estimate their size, like the Hausdorff dimension. We have adopted in **Chapter 10** a deeper approach based on a quantitative comparison in terms of the Hausdorff capacities $\mathcal{H}^s_\delta$. This is a first step towards the formalism of trace inequalities that is pursued later on in Chapters 15–17.

Different questions involving (DP) require different capacities. This issue is illustrated in Chapters 11–13 that focus on removable singularity problems. We begin in **Chapter 11** with Schwarz's prototype removable singularity principle for bounded harmonic functions and point singularities. We then tackle the general problem for various families of functions. The main tools concerning the equivalence of capacities and properties of the associated obstacle problems are developed in **Chapter 12**. We also revisit the Perron–Remak method as an obstacle problem. The characterization of removable sets is completed in **Chapter 13**, and is based on the explicit construction of solutions of (DP) for carefully chosen data μ.

In **Chapter 14**, we present a unified treatment to the question of strong approximation of diffuse measures. This is an important step towards the solution of the semilinear problem (SDP) for some given nonlinearity g. We follow an approach due to Mokobodzki [249] in the spirit of the Jordan decomposition theorem of a measure in terms of its positive and negative parts. The final answer is expressed in terms of a trace inequality.

In **Chapters 15–17** we review several aspects of trace inequalities, starting from the trace problem in Sobolev spaces, which leads one to the study of fractional spaces defined in terms of Gagliardo seminorms. We then present the Maz'ya–Adams formalism of trace inequalities, including the critical, purely geometric case related to the $W^{k,1}$ capacity for k integer.

Chapters 18 and 19 are devoted to measures which are diffuse with respect to the $W^{1,2}$ capacity. These measures are characterized in **Chapter 18** by the property that they are the strong limit of measures that yield continuous potentials. We also present an elegant functional characterization of the latter measures due to Aizenman and Simon [9]. In **Chapter 19**, we prove that (SDP) has a solution for every diffuse measure, regardless of the growth rate of g. This property completes the picture that is initiated in Chapter 4 and illustrates the universal role played by diffuse measures.

In **Chapter 20**, we take a closer look at the meaning of the zero boundary condition in the weak formulation of the (DP), by showing that it is equivalent to a zero average condition. We also explain why the method of sub- and supersolutions holds for (SDP) under the assumption that g is merely a continuous function, while it is commonplace to assume that g is Lipschitz continuous. These properties provide one with the existence of extremal solutions of (SDP) in the context of the Perron–Remak method. The existence of such solutions is used in **Chapter 21** to characterize measures for which the semilinear Dirichlet problem has a solution when g has power or exponential growths at infinity.

Chapter 22 is devoted to the strong maximum principle for nonnegative supersolutions u associated to the Schrödinger operator $-\Delta+V$, where the potential V belongs to $L^p(\Omega)$ for some exponent $1 \leqslant p \leqslant +\infty$. We prove that the possible size of the set $\{u = 0\}$ depends on the exponent p, and can be expressed in terms of a capacity. This last chapter beautifully illustrates how the tools we develop in this book can be implemented to investigate new properties of the Schrödinger operator.

The reader will find in **Appendices A and B** the definitions and main properties of the Sobolev and Hausdorff capacities that are used here. The exercises provide some complementary material, and are not necessarily intended to be solved in a first reading; their solutions can be found in **Appendix C**.

This project has originated from a set of lectures given at the Universidade Estadual de Campinas in 2005, and then from a full one-semester course in 2008. They were influenced by the enthusiasm and captivating style of H. Brezis, who had introduced me to these problems. In 2012, I deeply rewrote these notes, and the resulting monograph [286] won the Concours annuel in Mathematics of the Académie royale de Belgique. The text was then enlarged, including removable singularity principles and the Maz'ya–Adams trace inequalities. The notion of reduced measure, introduced with Brezis and Marcus [59] and pursued in [286], has been incorporated in the formalism of the nonlinear Perron-Remak method.

An updated list of corrections and misprints is available in my personal website at uclouvain.be, and is based on an interface that has been kindly developed by Y. Voglaire. The reader will find in the literature some recent advances on problems involving domains with little regularity [174, 234, 235], quasilinear operators in Euclidean spaces [163, 208, 216] or in metric spaces [28], Dirichlet problems involving trace measures on the boundary [224], connections to probability [118, 119, 198, 215], and trace inequalities [232, 237], that are not covered in this book.

I am indebted to several colleagues who have helped me with their comments and support: A. Ancona, L. Boccardo, P. Bousquet, D. de Figueiredo, Th. De Pauw, B. Devyver, L. Dupaigne, A. Farina, J. Mawhin, P. Mironescu, M. Montenegro, L. Moonens, H. Nguyen Quoc, F. Petitta, M. Pimenta, A. Presoto, J.-M. Rakotoson, P. Roselli and J. Van Schaftingen. Some of my students, in particular J. Dekeyser and N. Wilmet, have carefully read several chapters. I especially thank H. Brezis, L. Orsina, D. Spector, L. Véron and M. Willem for their advices, Th. Hinterman for his patience while waiting for the final version of the manuscript and M. Zunino for the typesetting. Partial financial support has been provided by the Fonds de la Recherche scientifique – FNRS from Belgium.

I am deeply grateful to my wife Isabelle for her encouragement. She also bravely kept Clément and Raphaël away from my computer while I was preparing the manuscript.

Louvain-la-Neuve, December 2016 Augusto C. Ponce